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LETTER TO THE EDITOR

Quantum groups and Lie-admissible time evolution

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Abstract. The time evolution of operators for q-oscillators is derived for the first time by exploiting the connection between q-deformation algebras and Lie-admissible algebras.

In recent years a great deal of attention has been paid both in the mathematical [1] and physical [2] literature to the so-called (improperly) quantum groups, i.e. 'deformations' of Lie algebras, first introduced by Arik and Coon [3] and later rediscovered independently by Kuryshkin [4], and by Faddeev [5], Sklyanin [6], and Kulish and Reshetiklin [7], in the study of the Yang–Baxter equations. Since 1981, many aspects of the q-deformation of an oscillator algebra have been investigated by Jannussis and collaborators [8–10].

In particular, Jannussis *et al* realized (as early as 1981) [8] that the standard form of the commutation relation for a q-deformed harmonic oscillator [3, 4]

$$\hat{A}\hat{A}^{+} - q\hat{A}^{+}\hat{A} = \hat{I} \tag{1}$$

(where $q \in [-1, \infty)$, $q \neq 0$) corresponds to a (λ, μ) mutation algebra [11], i.e. a special case of a Lie-admissible algebra [12]. A Lie-admissible Q-algebra [10] is obtained when considering an operator $\hat{Q}$ (instead of a number q) in the commutation relation, thus getting

$$(\hat{A}, \hat{A}^{+}) = \hat{A}\hat{A}^{+} - \hat{A}^{+}\hat{Q}\hat{A} = \hat{I}. \tag{2}$$

The connection between quantum groups and Lie-admissible Q-algebras has been extensively studied in [10]. Moreover, Jannussis and collaborators introduced the generalized commutation relation [10]

$$\hat{A}\hat{A}^{+} - \hat{A}^{+}\hat{Q}\hat{A} = f(\hat{n}) \tag{3}$$

where $\hat{n}$ is the usual number operator ($\hat{n}|n\rangle = n|n\rangle$) satisfying the following commutation rules with $\hat{A}$, $\hat{A}^{+}$:

$$[\hat{A}, \hat{n}] = \hat{A} \qquad [\hat{A}^{+}, \hat{n}] = -\hat{A}^{+} \tag{4}$$

and $f(\cdot)$ is a suitable function. For instance, in the case $f(\hat{n}) = q^{\hat{n}}$ (and q, Q numbers), one gets the Q-algebra depending on two parameters q, Q. For $Q = 1/q$ the q-deformed harmonic oscillator is obtained. These two special cases correspond to the quantum groups $SU(2)_{Q,q}$ and $SU(2)_q$, respectively [10, 2].

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In the present letter, we want to discuss the time evolution of the operators $\hat{A}^{+}(t)$ for the two-parameter Lie-admissible Q-algebra. As is known from the current literature [1-7], the analogous problem has not yet been solved for any quantum group, even for the case of the q-deformed oscillator.

In the Lie-admissible framework, the time evolution of operators is ruled by Santilli's generalization of Heisenberg's equation of motion [13], i.e.

$$\mathrm{i}\hbar\frac{\partial\hat{A}}{\partial t}=\hat{\mathscr{H}}\hat{T}\hat{A}-\hat{A}\hat{R}\hat{\mathscr{H}} \tag{5}$$

where $\hat{\mathscr{H}}$ is the usual Hamiltonian operator (describing conservative forces) and $\hat{T}$, $\hat{R}$ are suitable operators (supposed to represent, in general, non-conservative interactions). The case $\hat{T}=\hat{R}$ and $\hat{\mathscr{H}}\hat{T}\neq\hat{T}\hat{\mathscr{H}}$ corresponds to the so-called Lie-isotopic case.

According to [10], the boson realization of the operators $\hat{A}$ and $\hat{A}^{+}$ for $f(\hat{n})=q^{\hat{n}}$ (with Q a parameter) has the form

$$\hat{A}=\sqrt{\frac{[\hat{n}+1]_{Q,q}}{\hat{n}+1}}\,\hat{a}\qquad\hat{A}^{+}=\hat{a}^{+}\sqrt{\frac{[\hat{n}+1]_{Q,q}}{\hat{n}+1}} \tag{6}$$

where

$$[x]=\frac{q^{x}-Q^{x}}{q-Q}=\frac{Q^{x}-q^{x}}{Q-q} \tag{7}$$

and a, a^{+}, $\hat{n}=\hat{a}^{+}\hat{a}$ are boson operators, satisfying the usual commutation relations. Since the operators $\hat{A}^{+}\hat{A}$ and $\hat{a}^{+}\hat{a}$ commute, they have the same basis $|n\rangle$ (i.e. the usual Bose basis in Fock space).

From (6) one gets easily

$$\hat{A}|n\rangle=\sqrt{[n]_{Q,q}}|n-1\rangle\qquad\hat{A}^{+}|n\rangle=\sqrt{[n+1]_{Q,q}}|n+1\rangle \tag{8}$$

$$\hat{A}^{+}\hat{A}|n\rangle=[n]_{Q,q}|n\rangle \tag{9}$$

and the relations

$$\hat{A}\hat{A}^{+}=[\hat{n}+1]_{Q,q}\qquad\hat{A}^{+}\hat{A}=[\hat{n}]_{Q,q}. \tag{10}$$

The operators $\hat{A}$, $\hat{A}^{+}$ can be expressed in terms of the coordinate and momentum operators $\hat{x}$, $\hat{p}$ as [9]:

$$\begin{aligned}\hat{A}&=\sqrt{\frac{1}{\hbar(Q+1)}}\,\lambda\left(\sqrt{m\omega}\,\hat{x}+\mathrm{i}\frac{\hat{p}}{\sqrt{m\omega}}\right)\\ \hat{A}^{+}&=\sqrt{\frac{1}{\hbar(Q+1)}}\,\lambda\left(\sqrt{m\omega}\,\hat{x}-\mathrm{i}\frac{\hat{p}}{\sqrt{m\omega}}\right)\end{aligned} \tag{11}$$

where λ is a scale factor depending on q and Q.

From (11) and (2) (with $f(\hat{n})=q^{\hat{n}}$), we find the following non-canonical commutation relations for $\hat{x}$ and $\hat{p}$:

$$[\hat{x},\hat{p}]=\mathrm{i}\hbar\left(q^{\hat{n}}\lambda^{-2}+\frac{2\mathrm{i}}{\hbar\omega}\frac{Q-1}{Q+1}\hat{\mathscr{H}}\right) \tag{12}$$

where $\hat{\mathscr{H}}$ is the harmonic oscillator Hamiltonian

$$\hat{\mathscr{H}}=\frac{p^{2}}{2m}+\frac{m\omega^{2}}{2}\hat{x}^{2}=\frac{\hbar\omega}{4}(Q+1)\lambda^{-2}(\hat{A}\hat{A}^{+}+\hat{A}^{+}\hat{A}). \tag{13}$$

On account of (10) we get

$$\hat{\mathscr{H}}=\frac{\hbar\omega}{4}(Q+1)\lambda^{-2}([\hat{n}+1]_{Q,q}+[\hat{n}]_{Q,q}) \tag{14}$$

i.e. $\hat{\mathscr{H}}$ is a function of the number operator $\hat{n}$.

The exponential form of the time evolution for $\hat{A}$, obtained from the generalized Heisenberg equation (5) for the Lie-admissible Q-algebra (3), is given by [13] (we have of course, in this case, $\hat{T}=\hat{I}$ and $\hat{R}=\hat{Q}=Q\hat{I}$)

$$\hat{A}(t)=\exp\left(\frac{\mathrm{i}t\hat{\mathscr{H}}Q}{\hbar}\right)\hat{A}(0)\exp\left(\frac{-\mathrm{i}t\hat{\mathscr{H}}}{\hbar}\right). \tag{15}$$

On account of (6) and (14), $\hat{A}(t)$ becomes

$$\begin{aligned}\hat{A}(t)&=\exp\left\{\frac{\mathrm{i}t\omega(Q+1)Q}{4}\lambda^{-2}([\hat{n}+1]_{Q,q}+[\hat{n}]_{Q,q})\right\}\\&\quad\times\sqrt{\frac{[\hat{n}+1]_{Q,q}}{\hat{n}+1}}\,\hat{a}\exp\left\{\frac{-\mathrm{i}t\omega(Q+1)}{4}\lambda^{-2}([\hat{n}+1]_{Q,q}+[\hat{n}]_{Q,q})\right\}\\&=\exp\left\{\frac{\mathrm{i}t\omega(Q+1)}{4}\lambda^{-2}[Q([\hat{n}+1]_{Q,q}+[\hat{n}]_{Q,q})\right.\\&\quad\left.-([\hat{n}+2]_{Q,q}+[\hat{n}+1]_{Q,q})]\right\}\hat{A}(0).\end{aligned} \tag{16}$$

After some algebra we obtain finally

$$\hat{A}(t)=\exp\left[\frac{-\mathrm{i}t\omega(Q+1)(q+1)}{4\lambda^2}q^{\hat{n}}\right]\hat{A}(0) \tag{17}$$

and the corresponding expression for $\hat{A}^+$:

$$\hat{A}^+(t)=\hat{A}^+(0)\exp\left[\frac{\mathrm{i}t\omega(Q+1)(q+1)}{4\lambda^2}q^{\hat{n}}\right]. \tag{18}$$

The case $q=1$ (already discussed in [10]) corresponds to $\lambda=1$, and therefore

$$\hat{A}(t)=\mathrm{e}^{-\mathrm{i}t\omega(Q+1)/2}\hat{A}(0). \tag{19}$$

As already noted, the q-deformed oscillator is obtained for $Q=1/q$ ($\lambda=q^{-1/4}$), namely

$$\hat{A}(t)=\exp\left[-\mathrm{i}t\omega\frac{(q+1)^2}{4q^{1/2}}q^{\hat{n}}\right]\hat{A}(0). \tag{20}$$

In the same way, without any difficulty, it is possible to find the time evolution for $\hat{A}$, $\hat{A}^+$ for any (regular) function $f(\hat{n})$.

We have therefore shown that the formalism of Lie-admissible algebras allows one to discuss dynamics for quantum groups in a straightforward way. Further applications of this approach will be given elsewhere.

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